Lessons 24-26: Optimization

How to Solve Optimization Problems

1. Sketch and label a picture (if possible).
2. Write the equation that is to be maximized or minimized (the **objective function**).
3. Find the domain of the objective function.
4. Reduce the objective function to ONE independent variable using the **constraint equations**.
5. Use calculus to find the maximum or minimum.
6. Answer the question.

1. Find two positive numbers such that the first number plus twice the second number is a minimum and the product of the two numbers is 36.

(1) x = 1st number, y = 2nd number

(2) minimize $f(x,y) = x + 2y$

(3) $x, y > 0$

(4) Constraint: $xy = 36$

Solve for y: $y = \frac{36}{x}$

Rewrite f: $f(x) = x + 2\left(\frac{36}{x}\right)$

$= x + 72x^{-1}$

(5) $f'(x) = 1 - 72x^{-2}$

$= 1 - \frac{72}{x^2}$

$= \frac{x^2}{x^2} - \frac{72}{x^2}$

$= \frac{x^2 - 72}{x^2}$

CV's: $x^2 - 72 = 0$, $x^2 = 0$

$x = \sqrt{72}$, $\cancel{x = -\sqrt{72}}$, $\cancel{x = 0}$ ← not in domain

1st Derivative Test:

f': $-$, $+$

x: $\cancel{0}$, $\sqrt{72}$ ← absolute min

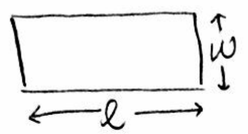
(6) $x = \sqrt{72} = \sqrt{3^2 \cdot 2^3} = 3 \cdot 2\sqrt{2} = 6\sqrt{2}$

$y = \frac{36}{6\sqrt{2}} = \frac{6}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$

$\boxed{x = 6\sqrt{2},\ y = 3\sqrt{2}}$

2. Find the dimensions of a rectangle with area 12cm^2 and a minimum perimeter. (We expect a square!)

①



② Minimize $P = 2l + 2w$

③ $l, w > 0$

④ Constraint: $lw = 12$

Solve for l: $l = \frac{12}{w}$

Rewrite P: $P = 2\left(\frac{12}{w}\right) + 2w$

$= 24w^{-1} + 2w$

⑤ $P' = -24w^{-2} + 2$

$= \frac{-24}{w^2} + 2$

$= \frac{-24 + 2w^2}{w^2}$

CV's: $-24 + 2w^2 = 0$, $w^2 = 0$ not in domain

$w = \sqrt{12}$, $\cancel{w = -\sqrt{12}}$, $\cancel{w = 0}$

1st Derivative Test:

P': $-$, $+$; w: 0, $\sqrt{12}$ absolute min

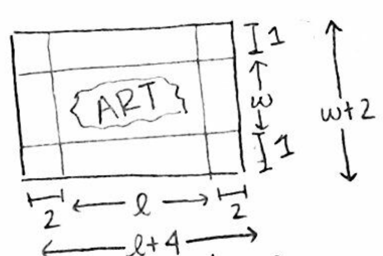
⑥ $w = \sqrt{12} = \sqrt{2^2 \cdot 3} = 2\sqrt{3}$

$l = \frac{12}{2\sqrt{3}} = \frac{6}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$

$\boxed{2\sqrt{3}\text{ cm} \times 2\sqrt{3}\text{ cm}}$

3. Ada wants to create a piece of art using a canvas with area 18ft^2. She wants to leave a one foot margin between the edge the canvas and the artwork on top and bottom, and two feet on either side. What dimensions of the canvas maximize the painted area?

①



② Maximize $A = lw$

③ $l, w > 0$

④ Constraint: $(w+2)(l+4) = 18$

Solve for l: $l + 4 = \frac{18}{w+2}$

$l = \frac{18}{w+2} - 4$

Rewrite A: $A = \left(\frac{18}{w+2} - 4\right)w$

$= \frac{18w}{w+2} - 4w$

⑤ $A' = \frac{(w+2)(18) - 18w(1)}{(w+2)^2} - 4$

$= \frac{\cancel{18w} + 36 - \cancel{18w}}{(w+2)^2} - 4$

$= \frac{36 - 4(w+2)^2}{(w+2)^2}$

CV's: $36 - 4(w+2)^2 = 0$, $(w+2)^2 = 0$

$(w+2)^2 = 9$ $\quad w + 2 = 0$

$w + 2 = 3$, $w + 2 = -3$ $\quad \cancel{w = -2}$

$w = 1$, $\cancel{w = -5}$

1st Derivative Test:

A': $-$, $+$; w: 0, 1 absolute min

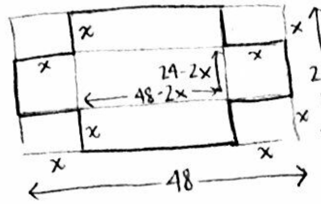
⑥ $w = 1$, $l = \frac{18}{1+2} - 4 = 2$

Dimensions of canvas:

$\boxed{6\text{ft} \times 3\text{ft}}$

4. A piece of cardboard is 24 inches by 48 inches. A square is to be cut from each corner and the sides folded up to make an open-top box. What is the maximum possible volume of the box?

1



2 Maximize $V = lwh$

3 $0 < l < 48,\ 0 < w < 24,\ 0 < h < 12$

4 $l = 48 - 2x,\ w = 24 - 2x,\ h = x$

Rewrite V: $V = (48-2x)(24-2x)(x)$

$= (48-2x)(24x - 2x^2)$

5 $V' = (-2)(24x - 2x^2) + (48-2x)(24-4x)$

$= -48x + 4x^2 + 1152 - 192x - 48x + 8x^2$

$= 12x^2 - 288x + 1152 \overset{\text{set}}{=} 0$

$12(x^2 - 24x + 96) = 0$

CV's: $x = \dfrac{24 \pm \sqrt{24^2 - 4(96)}}{2}$

$= \dfrac{24 \pm \sqrt{192}}{2} = \dfrac{24 \pm \sqrt{2^6 \cdot 3}}{2}$

$= \dfrac{24 \pm 8\sqrt{3}}{2} = 12 \pm 4\sqrt{3}$

CV: $x = 12 - 4\sqrt{3}$ ($12 + 4\sqrt{3}$ not in domain)

2nd Derivative Test:

$V'' = 12(2x - 24) < 0$ when $x = 12 - 4\sqrt{3}$

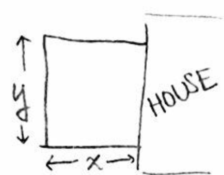
6 $V = (48 - 2(12 - 4\sqrt{3}))(24 - 2(12 - 4\sqrt{3}))(12 - 4\sqrt{3})$

$\approx \boxed{2{,}660.43 \text{ in}^3}$

5. You have 8L feet of fence to make a rectangular exercise pen for your dog alongside the wall of your house. (L is a positive constant.) The wall of the house will bound one side of the pen. What is the largest possible area of the pen?

L is a constant!

1



2 Maximize $A = xy$

3 $x, y > 0$ (and $y < 8L, x < 4L \ldots$)

4 Constraint: $2x + y = 8L$

Solve for y: $y = 8L - 2x$

Rewrite A: $A = x(8L - 2x)$

5 $A' = 1(8L - 2x) + x(-2)$

$= 8L - 2x - 2x$

$= 8L - 4x \overset{\text{set}}{=} 0$

CV's: $4x = 8L$

$x = 2L$

2nd Derivative Test:

$A'' = -4 < 0$

6 $A = 2L(8L - 2(2L))$

$= 2L(8L - 4L)$

$= 2L(4L)$

$= \boxed{8L^2 \text{ ft}^2}$